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A note on the space of lagrangian submanifolds of a symplectic 4-manifold

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Abstract

We show that the space of compact lagrangian submanifolds of a symplectic 4-manifold is a coisotropic submanifold of the space of all codimension two submanifolds, the latter being equipped with a natural symplectic structure. The characteristic foliation of this coisotropic submanifold is shown to coincide with the isodrastic foliation of Weinstein. We also show that the space of lagrangian submanifolds diffeomorphic to the 2-sphere is a lagrangian submanifold. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

This paper is a part of a general programme to investigate the geometry of the space $\mathrm{Sub}(N)$ of smooth submanifolds of a given finite-dimensional smooth manifold N. In general, the infinite-dimensional manifold $\mathrm{Sub}(N)$ has many connected components, each of which consists of submanifolds of N of a particular diffeomorphism type. For a smooth manifold M, we denote $\mathrm{Sub}(M, N)$ the submanifold of $\mathrm{Sub}(N)$ consisting of all submanifolds of N of diffeomorphism type M (so that $\mathrm{Sub}(M, N)$ is a union of components of $\mathrm{Sub}(N)$). Throughout this paper we assume that M, N are connected and without boundary, and M is compact. Having chosen N, we consider only those manifolds M

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for which $\mathrm{Sub}(M, N)$ is nonempty. It is often the case that providing M and N with geometric structures gives rise to some geometric structure on the manifold $\mathrm{Sub}(M, N)$, and we wish to know to what extent the three geometries interact. An interesting example arises when N is equipped with a volume element and when the codimension is equal to 2: a generalization of a construction of Marsden and Weinstein [7] then provides $\mathrm{Sub}(M, N)$ with a (weak) symplectic structure λ (see Section 3 for details). For $M = S^1$ and the volume element coming from a riemannian metric on a 3-manifold N, Brylinski [1] has shown that $(\mathrm{Sub}(M, N), \lambda)$ has a natural Kähler structure. LeBrun [6] has generalized Brylinski's ideas by considering the Kähler geometry of the space of codimension two timelike submanifolds in a pseudoriemannian manifold of arbitrary dimension. Another case, of key importance in general relativity theory, is the space of codimension two spacelike surfaces in a lorentzian manifold. It may be demonstrated that this space admits a natural parakählerian structure (unpublished work by the author). Imposing structure on M as well as on N leads to other interesting possibilities. For example, the choice of a volume element on M and a symplectic structure on N gives a symplectic structure on the space of smooth maps $C(M, N)$, the geometry of which has been studied by Donaldson [2], who develops several very interesting examples in the context of Kähler and hyperkähler geometry, and by Hitchin [5], who explores the structure of the space of special lagrangian submanifolds of a Calabi–Yau manifold.

Here, and in subsequent work [9], we examine the codimension two "pure" symplectic case, i.e., we assume that N is an arbitrary symplectic manifold with symplectic form ω, and that M has no structure save for a choice of orientation. The basic question is now the following: how does the symplectic geometry of $(\mathrm{Sub}(M, N), \lambda)$ interact with that of (N, ω)? In particular, how do the natural classes of submanifolds of (N, ω), e.g., symplectic submanifolds and coisotropic submanifolds, sit inside $(\mathrm{Sub}(M, N), \lambda)$? In this paper we focus on the first nontrivial case, i.e., when the dimension of N is equal to 4 and M is a surface, thereby commencing a study of the space of lagrangian submanifolds — a key theme in symplectic geometry — in an infinite-dimensional symplectic framework. Our main result is that the space $\mathrm{Sub}_0(M, N, \omega)$ of lagrangian submanifolds of (N, ω) of diffeomorphism type M is a coisotropic submanifold of $(\mathrm{Sub}(M, N), \lambda)$, and, in the special case $M = S^2$, $\mathrm{Sub}_0(M, N, \omega)$ is actually a lagrangian submanifold of $(\mathrm{Sub}(M, N), \lambda)$. In addition, the characteristic foliation of the coisotropic submanifold $\mathrm{Sub}_0(M, N, \omega)$ is shown to coincide with the isodrastic foliation of Weinstein [11] which he introduced to study the classical limit of the Berry phase.

The paper is organized as follows. In Section 2 we summarize standard material relating to differential forms on spaces of maps and on the manifold of submanifolds. This section also serves to introduce our notation. In Section 3 we specialize to the codimension two situation, and we define the symplectic form on the manifold of submanifolds of a symplectic manifold, as well as summarizing its basic invariance properties. Some general remarks on the space of lagrangian submanifolds are made in Section 4, and, in Section 5, we state and prove our main results for the four-dimensional case. Finally, we describe some possibilities for further work.

In what follows, all finite-dimensional manifolds and maps between them are smooth. If an appropriate category of Fréchet manifolds, e.g., the Nash–Moser category (see [4]) is chosen, then the same will be true also of all our infinite-dimensional manifolds and maps, but we do not discuss the analytical details here.

2. General framework

Let M, N be connected manifolds without boundary (of dimensions m and n, respectively), and assume that M is compact and oriented. Consider the manifold $C(M, N)$ of maps from M into N. For $f \in C(M, N)$, the tangent space is given by $T_f C(M, N) = \mathrm{Vect}_f(N) = \{Z \in C(M, TN) : \tau_N \circ Z = f\}$, where $\tau_N : TN \to N$ is the tangent bundle of N. We denote the natural actions of $\mathrm{Diff}(M)$ and $\mathrm{Diff}(N)$ on $C(M, N)$ by α and β, respectively. Thus, for $\phi \in \mathrm{Diff}(M)$, $\alpha_\phi \in \mathrm{Diff}(C(M, N))$ is given by $\alpha_\phi(f) = f \circ \phi$, and, for $\psi \in \mathrm{Diff}(N)$, $\beta_\psi \in \mathrm{Diff}(C(M, N))$ is given by $\beta_\psi(f) = \psi \circ f$ for all $f \in C(M, N)$. Note that α is a right action and β is a left action. We have the projections $p : C(M, N) \times M \to C(M, N); (f, x) \mapsto f$ and $q : C(M, N) \times M \to M; (f, x) \mapsto x$, and the evaluation map $e : C(M, N) \times M \to N; (f, x) \mapsto f(x)$. We regard p as a (trivial) fibration, so that we have the fibre integral $\int_p : \Omega(C(M, N) \times M) \to \Omega(C(M, N))$, using which we may construct differential forms on $C(M, N)$ in the standard manner (see, e.g., [1]): define $\Lambda : \Omega(M) \times \Omega(N) \to \Omega(C(M, N))$ by $\Lambda(\xi, \eta) = \int_p q^*\xi \wedge e^*\eta$ for all $\xi \in \Omega(M)$, $\eta \in \Omega(N)$. Note that if $\xi \in \Omega^a(M)$ and $\eta \in \Omega^b(M)$, then $\Lambda(\xi, \eta) \in \Omega^{a+b-m}(C(M, N))$ provided that $a + b \geq m$.

Using basic properties of the fibre integral, it is straightforward to prove the following:

Lemma 2.1.
1. $\Lambda(\phi^*\xi, \eta) = \alpha^*_{\phi^{-1}}(\Lambda(\xi, \eta))$ *for all* $\xi \in \Omega(M), \eta \in \Omega(N), \phi \in \mathrm{Diff}^+(M)$,
2. $\Lambda(\xi, \psi^*\eta) = \beta^*_\psi(\Lambda(\xi, \eta))$ *for all* $\xi \in \Omega(M), \eta \in \Omega(N), \psi \in \mathrm{Diff}(N)$.

Now consider the space of embeddings $\mathrm{Emb}(M, N)$ of M into N which is an open submainfold of $C(M, N)$. We assume that $\mathrm{Emb}(M, N)$ is nonempty in what follows, so that, in particular, $\dim M \leq \dim N$. We denote the restrictions of the maps defined above by the same letters. Via the right action α, we have the principal $\mathrm{Diff}^+(M)$-bundle $\gamma : \mathrm{Emb}(M, N) \to \mathrm{Sub}(M, N);\ f \mapsto [f] = f(M)$, where $\mathrm{Sub}(M, N) = \mathrm{Emb}(M, N)/\mathrm{Diff}^+(M)$ is the manifold of (oriented) submanifolds of N of diffeomorphism type M. The principal bundle γ may be thought of as a nonlinear analogue of the projection of a Stiefel manifold onto the corresponding Grassmann manifold. The vertical distribution of γ is given by $V_f = \mathrm{Ker}\, D\gamma(f) = Df \circ \mathrm{Vect}(M) \subseteq \mathrm{Vect}_f(N) = T_f \mathrm{Emb}(M, N)$ for all $f \in \mathrm{Emb}(M, N)$. Here $\mathrm{Vect}(M)$, the space of vector fields on M, is regarded as the Lie algebra of $\mathrm{Diff}^+(M)$ (with bracket equal to the negative of the Lie bracket of vector fields, as usual). The base manifold tangent spaces are given by $T_{[f]}\mathrm{Sub}(M, N) = \Gamma(\nu_f)$, where ν_f is the normal bundle of f for all $f \in \mathrm{Emb}(M, N)$. Note that, since the actions α and β on $\mathrm{Emb}(M, N)$ commute, the action β embeds $\mathrm{Diff}(N)$ as a subgroup of the group of automorphisms of γ, and we have $\gamma \circ \beta_\psi = \bar{\beta}_\psi \circ \gamma$ for all $\psi \in \mathrm{Diff}(N)$, where $\bar{\beta}$ is the natural action of $\mathrm{Diff}(N)$ on

Sub(M, N). We refer the reader to Swift [8] and references therein for more details on the structure of the principal bundle γ, and for a corresponding treatment of the manifold of immersions. The latter is more complicated than the embeddings case because the quotient space of the manifold of immersions by the diffeomorphism group is not itself a smooth manifold due to the presence of singularities corresponding to symmetric immersions. A natural resolution of this singular space is described in [8].

3. Codimension two submanifolds

We now introduce a construction in the codimension two case which is a generalization and modification of an idea due to Marsden and Weinstein [7] and Brylinski [1]. Suppose that $n - m = 2$ and that N is equipped with a volume element η. Define $\tilde{\lambda} = \Lambda(1, \eta) \in \Omega^2(\mathrm{Emb}(M, N))$. Since $\mathrm{d}\eta = 0$ and exterior differentiation commutes with the fibre integral, we see that $\mathrm{d}\tilde{\lambda} = 0$, i.e., $\tilde{\lambda}$ is a presymplectic form on $\mathrm{Emb}(M, N)$. From the definition of the fibre integral, a short computation shows that $\tilde{\lambda}$ is given by

$$\tilde{\lambda}(f)(Z, W) = \int_M f^* i_W i_Z \eta, \tag{1}$$

where $f^* i_W i_Z \eta \in \Omega^m(M)$ is defined by

$$(f^*_W i_Z \eta)(x)(v_1, \ldots, v_m) = \eta(f(x))(Z(x), W(x), Df(x)v_1, \ldots, Df(x)v_m), \tag{2}$$

for all $v_1, \ldots, v_m \in T_x M$, $x \in M$, $Z, W \in T_f \mathrm{Emb}(M, N)$, $f \in \mathrm{Emb}(M, N)$. Other important properties of $\tilde{\lambda}$ are given in Proposition 3.1.

Proposition 3.1.
1. $\alpha^*_\phi \tilde{\lambda} = \tilde{\lambda}$ *for all* $\phi \in \mathrm{Diff}^+(M)$,
2. $\beta^*_\psi \tilde{\lambda} = \tilde{\lambda}$ *for all* $\psi \in \mathrm{Diff}(N, \eta)$,
3. $\mathrm{Ker}\, \tilde{\lambda} = \mathrm{Ker}\, D\gamma$.

Proof.
1. Let $\phi \in \mathrm{Diff}^+(M)$. We have $\alpha^*_\phi \tilde{\lambda} = \alpha^*_\phi(\Lambda(1, \eta)) = \Lambda(\phi_* 1, \eta)) = \Lambda(1, \eta) = \tilde{\lambda}$, where we have used (1) of Lemma 2.1.
2. Let $\psi \in \mathrm{Diff}(N, \eta)$. We have $\beta^*_\psi \tilde{\lambda} = \beta^*_\psi(\Lambda(1, \eta)) = \Lambda(1, \psi^* \eta) = \Lambda(1, \eta) = \tilde{\lambda}$, where we have used (2) of Lemma 2.1.
3. (3) Let $f \in \mathrm{Emb}(M, N)$.

First suppose $Z \in \mathrm{Ker}\, D\gamma(f)$, so there exists $X \in \mathrm{Vect}(M)$ with $Z = Df \circ X$. Let $W \in T_f \mathrm{Emb}(M, N)$, $x \in M$, $v_1, \ldots, v_m \in T_x M$. Then, by (2), $(f^* i_W i_Z \eta)(x)(v_1, \ldots, v_m) = \eta(f(x))(Df(x)X(x), W(x), Df(x)v_1, \ldots, Df(x)v_m) = 0$ because $\{Df(x)X(x), Df(x) v_1, \ldots, Df(x)v_m\}$ is a linearly dependent set. Hence, $Z \in \mathrm{Ker}\, \tilde{\lambda}(f)$, so $\mathrm{Ker}\, D\gamma(f) \subseteq \mathrm{Ker}\, \tilde{\lambda}(f)$.

Conversely, suppose $Z \in \mathrm{Ker}\, \tilde{\lambda}(f)$, so that $\tilde{\lambda}(f)(Z, W) = 0$ for all $W \in T_f \mathrm{Emb}(M, N)$. Choose a riemannian metric k on N such that $\mathrm{vol}(k) = \eta$ and denote by ν^k_f the geometric normal bundle of $f : M \to (N, k)$. Then there exist unique $X \in \mathrm{Vect}(M)$ and $Z^\perp \in \Gamma(\nu_f^k)$ such that $Z = Df \circ X + Z^\perp$. Assume that $Z^\perp \neq 0$, so there exists an open set U in M such that $Z^\perp|_U$ is nowhere vanishing. Without loss of generality, we may assume that $\nu_f^k|_U$ is trivializable. Now define $\hat{Z} = Z^\perp / \|Z^\perp\||_U$ and choose $\hat{W} \in \Gamma(\nu_f^k|_U)$ such that $(\hat{Z}, \hat{W})$ is an oriented orthonormal trivialization of $\nu_f^k|_U$. Finally, let a be a bump function on U and define $W \in T_f\mathrm{Emb}(M, N)$ by

$$W(x) = \begin{cases} a(x)\hat{W}(x), & x \in U \\ 0, & x \notin U, \text{ for all } x \in M. \end{cases}$$

Then we have $0 = \tilde{\lambda}(f)(Z, W) = \tilde{\lambda}(f)(Z^\perp, W) = \int_M f^* i_W i_{Z^\perp} \eta = \int_U \|Z^\perp\| a \,\mathrm{vol}\,(f^*k) > 0$, a contradiction. We conclude that $Z^\perp = 0$, so that $Z = Df \circ X \in \mathrm{Ker}\, D\gamma(f)$. Hence, $\mathrm{Ker}\,\tilde{\lambda}(f) \subseteq \mathrm{Ker}\, D\gamma(f)$. □

Corollary 3.2.
1. *There exists a unique symplectic form* λ *on* $\mathrm{Sub}(M, N)$ *such that* $\gamma^*\lambda = \tilde{\lambda}$, *i.e.,* $(\mathrm{Sub}(M, N), \lambda)$ *is the reduced symplectic manifold associated with the presymplectic manifold* $(\mathrm{Emb}(M, N), \tilde{\lambda})$.
2. *Via the action* $\bar{\beta}$, $\mathrm{Diff}(N, \eta)$ *acts by symplectomorphisms on the symplectic manifold* $(\mathrm{Sub}(M, N), \lambda)$, *i.e.,* $\bar{\beta}_\psi^*\lambda = \lambda$ *for all* $\psi \in \mathrm{Diff}(N, \eta)$.

Proof.
1. The facts that $\alpha_\phi^*\tilde{\lambda} = \tilde{\lambda}$ for all $\phi \in \mathrm{Diff}^+(M)$, and $\mathrm{Ker}\, D\gamma \subseteq \mathrm{Ker}\,\tilde{\lambda}$ imply that $\tilde{\lambda}$ is γ-basic, so descends to a unique 2-form λ on $\mathrm{Sub}(M, N)$ satisfying $\gamma^*\lambda = \tilde{\lambda}$. Moreover, $\tilde{\lambda}$ is closed because γ^* is injective and $\tilde{\lambda}$ is closed, and λ is nondegenerate because $\mathrm{Ker}\,\tilde{\lambda} \subseteq \mathrm{Ker}\, D\gamma$.
2. We have $\gamma^*(\bar{\beta}_\psi^*\lambda) = (\bar{\beta}_\psi \circ \gamma)^*\lambda = (\gamma \circ \beta_\psi)^*\lambda = \beta_\psi^*(\gamma^*\lambda) = \beta_\psi^*\tilde{\lambda} = \tilde{\lambda} = \gamma^*\lambda$. Hence, since γ^* is injective, $\bar{\beta}_\psi^*\lambda = \lambda$ for all $\psi \in \mathrm{Diff}(N, \eta)$.

□

Definition 3.3. Suppose that $n - m = 2$ and that N is equipped with a volume element η. Then $(\mathrm{Sub}(M, N), \lambda)$ is called the *symplectic manifold associated with* (M, N, η).

Remark 3.4. *To be precise, we should refer to* λ *as a weak symplectic form inasmuch as it is only weakly nondegenerate, i.e. for each* $[f] \in \mathrm{Sub}(M, N)$, *the linear map* $\bar{Z} \mapsto \lambda([f])(\bar{Z}, \cdot)$ *of* $T_{[f]}\mathrm{Sub}(M, N)$, *which is a Fréchet space, into* $T_{[f]}^*\mathrm{Sub}(M, N)$, *which is not a Fréchet space, is injective with dense image, but it is not surjective* (*see* [1,4]). *The adjective weak should always be understood, but it will be omitted in what follows.*

As remarked in Section 1, there are several important classical geometries that give rise to a volume element, so the above construction may be performed in a variety of interesting situations. In this paper, we are concerned solely with the symplectic case. Thus, suppose that ω is a symplectic form on N with $n = 2r \geq 4$, and let $\eta = \omega^r/r!$ be the corresponding

Liouville form. In this case, the presymplectic form $\tilde{\lambda}$ (and hence the symplectic form λ) may be written explicitly in terms of the symplectic form ω.

Lemma 3.5. *Let* $f \in \mathrm{Emb}(M, N)$, $Z, W \in T_f\mathrm{Emb}(M, N)$. *Then,*

$$\tilde{\lambda}(f)(Z, W) = \frac{1}{(r-1)!}\int_M ((\omega \circ f)(Z, W) f^*\omega - (r-1) f^* i_Z\omega \wedge f^* i_W \omega) \wedge (f^*\omega)^{r-2}.$$

Proof. A simple computation using Eq. (1). □

Proposition 3.6. *Via the action* $\bar{\beta}$, $\mathrm{Diff}(N, \omega)$, *the symplectomorphism group of* (N, ω), *acts by symplectomorphisms on the symplectic manifold* $(\mathrm{Sub}(M, N), \lambda)$, *i.e.,* $\bar{\beta}^*_\psi \lambda = \lambda$ *for all* $\psi \in \mathrm{Diff}(N, \omega)$.

Proof. Noting that $\mathrm{Diff}(N, \omega) \leq \mathrm{Diff}(N, \eta)$, we just apply (2) of Corollary 3.2. □

The programme now is to investigate the interaction of the symplectic geometry of the finite-dimensional symplectic manifold (N, ω) with that of the infinite-dimensional symplectic manifold $(\mathrm{Sub}(M, N), \lambda)$. In particular, it is of interest to examine how special classes of submanifolds of (N, ω), e.g., symplectic submanifolds, lagrangian submanifolds, coisotropic submanifolds, fit together in $(\mathrm{Sub}(M, N), \lambda)$. The symplectic submanifolds case is analysed in [9], and in Section 5 we apply our framework to the space of lagrangian submanifolds of a symplectic 4-manifold. First, for convenience and by way of motivation, we make a few remarks on the space of lagrangian submanifolds in general.

4. The space of lagrangian submanifolds of a symplectic manifold

For the convenience of the reader, we summarize the local structure of the space of lagrangian submanifolds of an arbitrary symplectic manifold using the language of this paper. For the technical details, we refer the reader to [10].

Let (N, ω) be a symplectic manifold and M a compact, oriented manifold with $n = 2m$. Let $\mathrm{Emb}_0(M, N, \omega) = \{f \in \mathrm{Emb}(M, N) : f^*\omega = 0\}$ be the space of lagrangian embeddings of M into (N, ω), and $\mathrm{Sub}_0(M, N, \omega) = \mathrm{Emb}_0(M, N, \omega)/\mathrm{Diff}^+(M)$ the space of (oriented) lagrangian submanifolds of (N, ω) of diffeomorphism type M. Using the cotangent bundle charts given by the Weinstein lagrangian neighbourhood theorem [10], it can be shown that $\mathrm{Emb}_0(M, N, \omega)$ is a closed submanifold of $\mathrm{Emb}(M, N)$, and that $\mathrm{Sub}_0(M, N, \omega)$ is a closed submanifold of $\mathrm{Sub}(M, N)$. We then have the principal $\mathrm{Diff}^+(M)$-fibration $\gamma_0 : \mathrm{Emb}_0(M, N, \omega) \to \mathrm{Sub}_0(M, N, \omega)$, where $\gamma_0 = \gamma|_{\mathrm{Emb}_0(M,N,\omega)}$. We remark that the action β of the group $\mathrm{Diff}(N, \omega)$ by automorphisms of γ restricts to an action by automorphisms of the closed principal subbundle γ_0.

Now let $f \in \mathrm{Emb}_0(M, N, \omega)$. Differentiating the lagrangian condition $f^*\omega = 0$ gives us that $T_f\mathrm{Emb}_0(M, N, \omega) = \{Z \in \mathrm{Vect}_f(N) : \mathrm{d}(f^* i_Z\omega) = 0\} \subseteq \mathrm{Vect}_f(N) = T_f\mathrm{Emb}$

(M, N). Since f is lagrangian, the epimorphism $T_f\mathrm{Emb}(M, N) \to \Omega^1(M);\ Z \mapsto f^* i_Z\omega$ projects to a natural isomorphism $T_f\mathrm{Sub}(M, N) \to \Omega^1(M);\ Z + Df \circ \mathrm{Vect}(M) \mapsto f^* i_Z\omega$. Similarly, the epimorphism $T_{[f]}\mathrm{Emb}_0(M, N, \omega) \to Z^1(M);\ Z \mapsto f^* i_Z\omega$ projects to a natural isomorphism of $T_{[f]}\mathrm{Sub}_0(M, N, \omega)$ onto $Z^1(M)$. Here, $Z^1(M)$ is the space of closed 1-forms on M. Using these isomorphisms, we will identify $T_{[f]}\mathrm{Sub}(M, N)$ with $\Omega^1(M)$ and $T_{[f]}\mathrm{Sub}_0(M, N, \omega)$ with $Z^1(M)$ for all $[f] \in \mathrm{Sub}_0(M, N, \omega)$.

The above remarks may be summarized in the following manner. Let $i_0 : \mathrm{Sub}_0(M, N, \omega) \to \mathrm{Sub}(M, N)$ denote inclusion. Then the exact sequence $0 \to \tau_{\mathrm{Sub}_0(M,N,\omega)} \to i_0^* \tau_{\mathrm{Sub}(M,N)} \to \nu_{i_0} \to 0$ of vector bundles over $\mathrm{Sub}_0(M, N, \omega)$ is just the exact sequence of vector bundles associated to the principal $\mathrm{Diff}^+(M)$-bundle γ_0 via the pushforward action of $\mathrm{Diff}^+(M)$ on the exact sequence $0 \to Z^1(M) \to \Omega^1(M) \to \Omega^1(M)/Z^1(M) \to 0$ of real vector spaces.

Finally, we describe the natural foliation on the manifold $\mathrm{Sub}_0(M, N, \omega)$. Denote by $B^1(M)$ the space of exact 1-forms on M. For $f \in \mathrm{Emb}_0(M, N, \omega)$, define the subspace $\tilde{I}_f$ of $T_f\mathrm{Emb}_0(M, N, \omega)$ by $\tilde{I}_f = \{Z \in \mathrm{Vect}_f(N) : f^* i_Z\omega \in B^1(M)\}$, and denote the corresponding subspace of $T_{[f]}\mathrm{Sub}_0(M, N, \omega)$ by $I_{[f]}$. Then, as above, the epimorphism $\tilde{I}_f \to B^1(M); Z \mapsto f^* i_Z\omega$ projects to an isomorphism of $I_{[f]}$ onto $B^1(M)$. A distribution on $\mathrm{Sub}_0(M, N, \omega)$ is defined as follows: associate with $[f] \in \mathrm{Sub}_0(M, N, \omega)$ the subspace $I_{[f]}$ of $T_{[f]}\mathrm{Sub}_0(M, N, \omega)$. Note that the corresponding subbundle of $\tau_{\mathrm{Sub}_0(M,N,\omega)}$ may be identified with the vector bundle $\mathrm{Emb}_0(M, N, \omega) \times_{\mathrm{Diff}^+(M)} B^1(M) \to \mathrm{Sub}_0(M, N, \omega)$. Weinstein [11] has shown that this distribution is integrable and the corresponding foliation I is called the *isodrastic foliation* of $\mathrm{Sub}_0(M, N, \omega)$.

5. The space of lagrangian submanifolds of a symplectic 4-manifold

We now discuss the space of lagrangian submanifolds of a symplectic 4-manifold within the framework described in Section 3 (note that this is possible because $2 + 2 = 2 \times 2$). Let (N, ω) be a symplectic 4-manifold and M a compact, oriented surface. Then (Definition 3.3), putting $\eta = \frac{1}{2}(\omega \wedge \omega)$, we have the symplectic manifold $(\mathrm{Sub}(M, N), \lambda)$ associated with (M, N, η). From Lemma 3.5, the corresponding presymplectic form is given by the following proposition.

Proposition 5.1. *Let* $f \in \mathrm{Emb}(M, N)$, $Z, W \in T_f\mathrm{Emb}(M, N)$. *Then,*

$$\tilde{\lambda}(f)(Z, W) = \int_M ((\omega \circ f)(Z, W) f^*\omega - f^* i_Z\omega \wedge f^* i_W\omega). \tag{3}$$

Corollary 5.2. *Let* $[f] \in \mathrm{Sub}_0(M, N, \omega)$. Then the symplectic form on $T_{[f]}\mathrm{Sub}(M, N) = \Omega^1(M)$ *is given by*

$$\lambda([f])(\sigma, \tau) = \int_M \tau \wedge \sigma \tag{4}$$

for all $\sigma, \tau \in \Omega^1(M)$.

Proof. Use (3), the lagrangian condition, and the natural isomorphisms described in Section 4. □

In order to examine the nature of the embedding $i_0 : \mathrm{Sub}_0(M, N, \omega) \to \mathrm{Sub}(M, N)$ w.r.t. the symplectic structure λ on $\mathrm{Sub}(M, N)$, we must consider, for each $[f] \in \mathrm{Sub}_0(M, N, \omega)$, how the subspace $T_{[f]}\mathrm{Sub}_0(M, N, \omega)$ sits inside the symplectic vector space $(T_{[f]}\mathrm{Sub}(M, N), \lambda([f]))$. To do this, we compute the symplectic complement $(T_{[f]}\mathrm{Sub}_0(M, N, \omega))^{\lambda([f])}$.

Lemma 5.3. *Let* $[f] \in \mathrm{Sub}_0(M, N, \omega)$. *Then* $(T_{[f]}\mathrm{Sub}_0(M, N, \omega))^{\lambda([f])} = B^1(M)$.

Proof. First let $\sigma \in (T_{[f]}\mathrm{Sub}_0(M, N, \omega))^{\lambda([f])}$, so that $\lambda([f])(\sigma, \tau) = 0$ for all $\tau \in T_{[f]}\mathrm{Sub}_0(M, N, \omega) = Z^1(M)$. Hence, $\int_M b\, \mathrm{d}\sigma = -\int_M \mathrm{d}b \wedge \sigma = -\lambda([f])(\sigma, \mathrm{d}b) = 0$ for all $b \in \Omega^0(M)$. We conclude that $\mathrm{d}\sigma = 0$, i.e., $\sigma \in Z^1(M)$. Now consider $[\sigma] \in H^1(M)$, where $H^1(M)$ is the first de Rham cohomology space of M. By Poincaré duality, $H^1(M)$ is equipped with the symplectic form Ω_M, where $\Omega_M([\tau_1], [\tau_2]) = \int_M \tau_2 \wedge \tau_1$ for all $[\tau_1], [\tau_2] \in H^1(M)$. Suppose that $\tau \in Z^1(M)$. Then, $\Omega_M([\sigma], [\tau]) = \int_M \tau \wedge \sigma = \lambda([f])(\sigma, \tau) = 0$, so by the nondegeneracy of Ω_M, we have $[\sigma] = 0$, i.e., $\sigma \in B^1(M)$. We conclude that $(T_{[f]}\mathrm{Sub}_0(M, N, \omega))^{\lambda([f])} \subseteq B^1(M)$.

Conversely, suppose that $\sigma \in B^1(M)$, say $\sigma = \mathrm{d}a$, where $a \in \Omega^0(M)$. Let $\tau \in T_{[f]}\mathrm{Sub}_0(M, N, \omega) = Z^1(M)$. Then, $\lambda([f])(\sigma, \tau) = \int_M \tau \wedge \sigma = \int_M \tau \wedge \mathrm{d}a = \int_M a\, \mathrm{d}\tau = 0$. Hence, $\sigma \in (T_{[f]}\mathrm{Sub}_0(M, N, \omega))^{\lambda([f])}$, so $B^1(M) \subseteq (T_{[f]}\mathrm{Sub}_0(M, N, \omega))^{\lambda([f])}$. □

Theorem 5.4. *Let* (N, ω) *be a symplectic 4-manifold and* M *a compact, oriented surface. Consider the embedding* $i_0 : \mathrm{Sub}_0(M, N, \omega) \to (\mathrm{Sub}(M, N), \lambda)$ *of the manifold of lagrangian submanifolds of* (N, ω) *of diffeomorphism type* M *into the symplectic manifold of all submanifolds of* N *of diffeomorphism type* M. *Then,*
1. i_0 *is a coisotropic embedding,*
2. i_0 *is a lagrangian embedding if and only if* $M = S^2$.

Proof. Let $[f] \in \mathrm{Sub}_0(M, N, \omega)$. Then, by Lemma 5.3, $(T_{[f]}\mathrm{Sub}_0(M, N, \omega))^{\lambda([f])} = B^1(M) \subseteq Z^1(M) = T_{[f]}\mathrm{Sub}_0(M, N, \omega)$, so $T_{[f]}\mathrm{Sub}_0(M, N, \omega)$ is a coisotropic subspace of the symplectic vector space $(T_{[f]}\mathrm{Sub}(M, N), \lambda([f]))$. Hence, the inclusion $i_0 : \mathrm{Sub}_0(M, N, \omega) \to (\mathrm{Sub}(M, N), \lambda)$ is a coisotropic embedding. Furthermore, i_0 is a lagrangian embedding if and only if $B^1(M) = Z^1(M)$, i.e., if and only if $H^1(M) = 0$. Since $\dim H^1(M) = 2\,\mathrm{genus}(M)$, we see that i_0 is a lagrangian embedding if and only if $M = S^2$. □

Remark 5.5. *Consider the manifold* $\mathrm{Sub}(M, N)$. *This may be given extra structure in two different ways. On the one hand, choosing a volume element* η *on* N *furnishes* $\mathrm{Sub}(M, N)$ *with the associated symplectic structure* λ (*Definition* 3.3). *On the other hand, choosing a symplectic form* ω *on* N *gives rise to the closed submanifold* $\mathrm{Sub}_0(M, N, \omega)$ *of* $\mathrm{Sub}(M, N)$ *as in Section* 4. *Theorem* 5.4 *may now be interpreted in terms of the interaction between these* *two methods of imposing structure on* $\mathrm{Sub}(M, N)$*. Fix a volume element* η *on* N *and consider the symplectic forms on* N *whose Liouville form is equal to* η*; such symplectic forms will be called* η*-compatible. The key point is that the lagrangian submanifolds of each* η*-compatible symplectic form* ω *are 'detected' by the symplectic structure* λ *— which depends only on* η *— in the sense that each of the submanifolds* $\mathrm{Sub}_0(M, N, \omega)$ *is actually coisotropic in* $(\mathrm{Sub}(M, N), \lambda)$*. Thus, the compatibility of symplectic forms and volume elements manifests itself at the level of lagrangian submanifolds. This idea will be explored further elsewhere, as will the way in which the family of coisotropic submanifolds* $\{\mathrm{Sub}_0(M, N, \omega) : \omega$ *is* η*-compatible*$\}$ *fit together in the symplectic manifold* $(\mathrm{Sub}(M, N), \lambda)$ *associated with* η*.*

Let us return now to the situation where we have a fixed symplectic form ω on N, and consider the symplectic reduction associated with the coisotropic embedding $i_0 : \mathrm{Sub}_0(M, N, \omega) \to (\mathrm{Sub}(M, N), \lambda)$. Since $(T_{[f]}\mathrm{Sub}_0(M, N, \omega))^{\lambda([f])} = B^1(M) = I_{[f]}$ for each $[f] \in \mathrm{Sub}_0(M, N, \omega)$, the characteristic distribution of i_0 is precisely Weinstein's isodrastic distribution I as described in Section 4 (thus we have an alternative proof of the integrability of I in the four-dimensional case). This gives us a foliation of $\mathrm{Sub}_0(M, N, \omega)$ by isotropic submanifolds of $(\mathrm{Sub}(M, N), \lambda)$. Furthermore, locally at least, we have a fibration $\pi : \mathrm{Sub}_0(M, N, \omega) \to P$, where the leaf space $P = \mathrm{Sub}_0(M, N, \omega)/I$ is equipped with a symplectic structure Ω satisfying $\pi^*\Omega = i_0^*\lambda$ (the global existence of such a fibration will be explored elsewhere). We now give a more concrete description of (P, Ω), the reduced symplectic manifold associated with i_0. First observe that $\mathrm{Diff}^+(M)$ acts by linear symplectomorphisms on the symplectic vector space $(H^1(M), \Omega_M)$, where, as in the proof of Lemma 5.3, Ω_M is the symplectic form coming from Poincaré duality. Therefore, associated to the principal $\mathrm{Diff}^+(M)$-bundle γ_0 is the symplectic vector bundle $\nu : \mathrm{Emb}_0(M, N, \omega) \times_{\mathrm{Diff}^+(M)} H^1(M) \to \mathrm{Sub}_0(M, N, \omega)$. Using Corollary 5.2 and Lemma 5.3, we see that ν may be naturally identified with the symplectic vector bundle $\pi^*\tau_P = \tau_{\mathrm{Sub}_0(M,N,\omega)}/\tau^{\lambda}_{\mathrm{Sub}_0(M,N,\omega)}$. To summarize:

Theorem 5.6. *Let* (N, ω) *be a symplectic 4-manifold and* M *a compact, oriented surface. Then each tangent space of the reduced symplectic manifold* (P, Ω) *associated with the coisotropic embedding* $i_0 : \mathrm{Sub}_0(M, N, \omega) \to (\mathrm{Sub}(M, N), \lambda)$ *may be naturally identified with the finite-dimensional symplectic vector space* $(H^1(M), \Omega_M)$*.*

Finally we note that, in the case $M = S^2$, the reduced symplectic manifold is just a point. Furthermore, using the ideas in the penultimate paragraph of Section 4, we see that the total space of the cotangent bundle of the lagrangian submanifold $\mathrm{Sub}_0(M, N, \omega)$ of $(\mathrm{Sub}(M, N, \lambda))$ is given by $T^*\mathrm{Sub}_0(M, N, \omega) = \mathrm{Emb}_0(M, N, \omega) \times_{\mathrm{Diff}^+(M)} (\Omega^1(M)/Z^1(M))$.

6. Further work

In this paper we have examined some basic aspects of the space of lagrangian surfaces in a symplectic 4-manifold within a framework of infinite-dimensional symplectic geometry. Clearly there remains much to be done in terms of both theory and applications. Examples of interesting possibilities for further work include: further connections with Weinstein's approach to the Berry phase [11]; links with symplectic topological work on lagrangian knots, see, e.g. [3] and references therein; the action of $\mathrm{Diff}(N, \omega)$, and of the subgroup $\mathrm{Ham}(N, \omega)$ of hamiltonian symplectomorphisms, on $\mathrm{Sub}_0(M, N, \omega)$; the role of $\mathrm{Sub}_0(M, N, \omega)$ in the geometric quantization of the symplectic manifold $(\mathrm{Sub}(M, N), \lambda)$ in the case $M = S^2$; computations for particular symplectic 4-manifolds such as products of surfaces, cotangent bundles of surfaces, and complex surfaces. We intend to return to some of these areas elsewhere. We remark that some of these ideas are also relevant in the study of the manifold of symplectic submanifolds which may be analysed using a framework of infinite-dimensional symplectic fibrations, symplectic connections and weak coupling forms (see [9]).

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